A Numerical Absorbing Boundary Condition for Finite Difference and Finite Element Analysis of Open Periodic Structures

Amir Boag and Raj Mittra

***Abstract*—In this paper we present a novel approach to deriving local boundary conditions, that can be employed in conjunction with the Finite Difference/Finite Element Methods (FD/FEM) to solve electromagnetic scattering and radiation problems involving periodic structures. The key step in this approach is to derive linear relationships that link the value of the field at a boundary grid point to those at the neighboring points. These linear relationships are identically satisfied not only by all of the propagating Floquet modes but by a few of the leading evanescent ones as well. They can thus be used in lieu of absorbing boundary conditions (ABCs) in place of the usual FD/FEM equations for the boundary points. Guidelines for selecting the orders of the evanescent Floquet modes to be absorbed are given in the paper. The present approach not only provides a simple way to derive an accurate boundary condition for mesh truncation, but also preserves the banded structure of the FD/FEM matrices. The accuracy of the proposed method is verified by using an internal check and by comparing the numerical results with the analytic solution for perfectly conducting strip gratings.**

I. Introduction

Analysis of electromagnetic scattering and radiation by relatively simple periodic structures is conveniently formulated by using such techniques as the Method of Moments (MoM) [1], or the Current Model Method [2], where the radiation condition is inherently satisfied. However, the Finite Difference/Finite Element Methods (FD/FEM) are often preferred over the MoM approach for problems involving inhomogeneous materials or complex geometries. The use of these methods for periodic structures requires one to couple, explicitly, the FD/FEM computational domain to the unbounded free space external to the FD/FEM mesh region. The FD/FEM solution can either be matched to the Floquet mode expansion [3], or be combined with the MoM to satisfy the radiation condition [4]. Both ways lead to non-local boundary conditions that are exact, albeit at the expense of spoiling the sparsity of the matrices generated in these formulations.

The local type of boundary conditions conventionally employed in the FD/FEM analysis of non-periodic structures such as Bayliss-Turkel [5] and Engquist-Majda [6], are not very well-suited for periodic geometries. Typically, the periodic structures of interest have cell sizes that are either smaller than, or comparable to, the wavelength of the illuminating field. Thus, the field radiated or scattered by such a structure consists of a limited number of propagating and an infinite series of evanescent Floquet modes. For waveguide problems, Bayliss and Turkel [7] have proposed a boundary condition designed to absorb all of the propagating waveguide modes. At any finite distance away from the inhomogeneous region, only a typically small number of the dominant evanescent modes produce a non-negligible contribution. In this paper, we introduce a numerical absorbing boundary condition that effects a complete absorption of *all* of the significant Floquet modes impinging upon the boundary, including both the propagating and the dominant evanescent modes. Using this requirement as the specification, we go on to derive linear relationships that link the field values at the boundary grid points to those of their neighbors. This procedure differs from the classical approaches to deriving the local boundary conditions, since in this method only the field values at the prescribed grid points are used. In this respect, it is somewhat similar to the measured equation of invariance (MEI) boundary condition [8]. However, as in [7], and in contrast to the MEI method, the present method uses the Floquet modes which depend only upon the periodicity and the excitation, and not on the geometry of the periodic element. This has an important advantage of using simple and analytically-known functions, and eliminating the need to carry out the surface integrations needed in other approaches.

The two-dimensional (2D), transverse magnetic (TM) polarization case is presented in this paper to illustrate the application of the proposed method. Because of space limitations, the dual case of transverse electric (TE) polarization, as well as the extension to the three-dimensional (3D) cases, will be deferred until forthcoming publications.

II. Formulation

We consider a two-dimensional (2D) periodic structure whose geometry and the relevant coordinate system is depicted in Fig. 1. The structure is periodic along the x-axis, its period is d, and it is uniform along the y-axis. The excitation is TM (transverse-magnetic relative to the y-axis) and quasi-periodic, with a constant phase shift of $kd\sin\theta_0$ between neighboring unit cells. Here, k is the free space wavenumber, and θ_0 is the scan angle in the phased array problem and the angle of incidence in the grating scattering case. The harmonic time dependence $e^{j\omega t}$ is implicit. It is well-known

Manuscript received October 19, 1993; revised March 2, 1994.

The authors are with the Electromagnetic Communication Laboratory, Department of Electrical and Computer Engineering, University of Illinois, Urbana, IL 61801 USA.

IEEE Log Number 9406798.

0018–9480/95$4.00

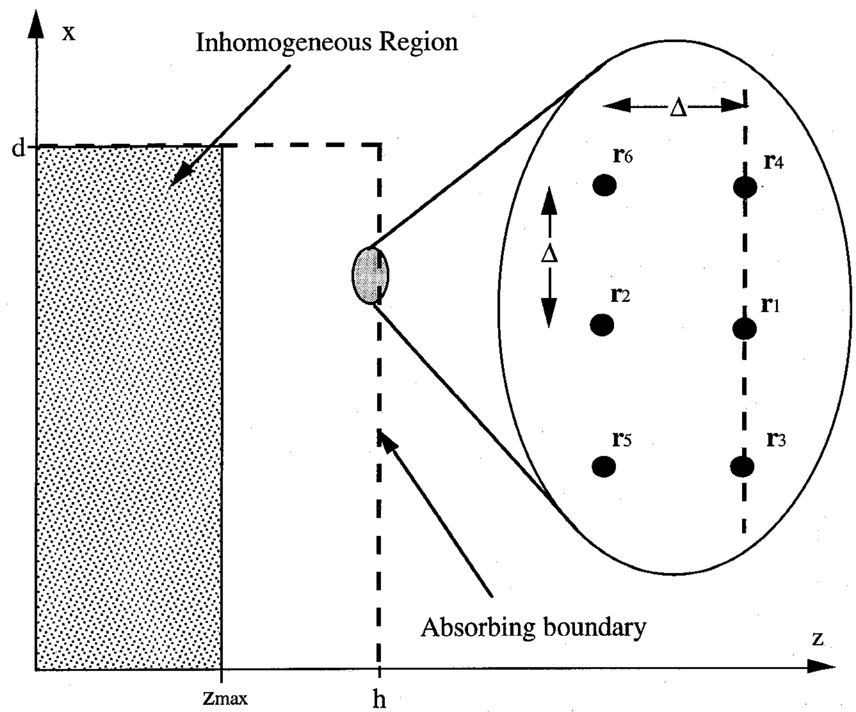


Fig. 1. The problem geometry and an arrangement of grid points for deriving the NABC.

that the problem domain can be reduced to a single unit cell by introducing the phase shift walls. We consider the problem of truncating the domain of discretization in the positive z-direction at a distance $z = h$. (Truncation of the domain in the negative z direction can be treated as a mirror image of the problem discussed in this Section.) It is assumed that all of the inhomogeneities are confined within the region $z < z_{\max}$. A procedure for truncating the mesh at the boundary $z = h$ by using a numerical absorbing boundary condition (NABC) will be presented below. The NABC provides the requisite equations in the FD/FEM formulation that serve to replace the corresponding equations at the grid points on the boundary $z = h$. The fields in the homogeneous region $z > z_{\max}$ can be expressed as a superposition of outgoing Floquet modes derivable from the y-component of the electric field E_y, which is given by

$$E_y = \sum_{n=-\infty}^{\infty} a_n E_{yn} \tag{1}$$

where

$$E_{yn} = e^{-j(k_{xn}x + k_{zn}z)} \tag{2}$$

is the electric field of the nth Floquet mode and a_n denotes its amplitude. In (2),

$$k_{xn} = k \sin\theta_0 + \frac{2\pi n}{d} \tag{3}$$

and

$$k_{zn} = \sqrt{k^2 - k_{xn}^2}, \quad \mathrm{Re}\{k_{zn}\} \geq 0, \quad \mathrm{Im}\{k_{zn}\} \leq 0 \tag{4}$$

are, respectively, the x and z components of the wavevector of the nth Floquet mode.

In the proposed procedure, we derive a linear relationship between the values of the field at a sets of grid points that are located on or near the truncation boundary. Consider a set of L points $\mathbf{r}_i$ with $i = 1, \ldots, L$. A possible arrangement of the points for the case of $L = 6$, that is suitable for a rectangular mesh, is shown in Fig. 1. Ideally, we would like to find the coefficients c_i, with $i = 1, \ldots, L$, such that they satisfy the relationship

$$\sum_{i=1}^{L} c_i E_y(\mathbf{r}_i) = 0 \tag{5}$$

Since in conventional phased arrays, and in frequency selective surface (FSS) applications, the period d is typically less than λ, the higher-order Floquet modes are expected to be highly evanescent. Thus, we can effectively limit our attention to the $L - 1$ modes of orders $n = N_1, \ldots, N_2$, where $N_2 - N_1 = L - 2$, that suffer from the least attenuation. The requirement that (5) be satisfied for these $L - 1$ modes can be expressed in the matrix form

$$\mathbf{U}\mathbf{c} = 0 \tag{6}$$

where $\mathbf{c}$ is an L-element column vector of the sought-for coefficients and

$$\mathbf{U} = \begin{bmatrix} E_{y,N_1}(\mathbf{r}_1) & \cdots & E_{y,N_1}(\mathbf{r}_L) \\ \vdots & \ddots & \vdots \\ E_{y,N_2}(\mathbf{r}_1) & \cdots & E_{y,N_2}(\mathbf{r}_L) \end{bmatrix} \tag{7}$$

is a $L - 1$ by L matrix of the values of the $L - 1$ selected modes at the L points. Clearly, one of the components of $\mathbf{c}$ can be chosen arbitrarily. Thus, if we let $c_1 = 1$, the solution for $\mathbf{c}$ can be written as

$$\mathbf{c} = \begin{bmatrix} 1 \\ -\mathbf{W}^{-1}\mathbf{v} \end{bmatrix} \quad (8)$$

where $\mathbf{v}$ is the first column of $\mathbf{U}$, and $\mathbf{W}$ is a square $L-1$ by $L-1$ matrix containing $L-1$ last columns of $\mathbf{U}$.

In general, this procedure can be repeated for every point on the boundary. If the grid in the layer adjacent to the boundary is uniform, the same coefficients $\mathbf{c}$ can be used for all the boundary points. In the uniform grid case, the NABC can be further improved by using the 'discrete' Floquet modes with the corrected dispersion relation

$$k_{zn} = \frac{1}{\Delta} a\cos\left(2 - (k\Delta)^2/2 - \cos(k_{xn}\Delta)\right), \quad \mathrm{Re}\{k_{zn}\} \geq 0, \quad \mathrm{Im}\{k_{zn}\} \leq 0 \quad (9)$$

in place of (4), where Δ denotes the mesh size. With this modification, a complete absorption of all of the $L-1$ Floquet modes under consideration will be guaranteed. Note that dispersion relation (9) was derived by substituting Floquet mode expression (2) into the central difference approximation of the Helmholtz equation. It can be easily modified to accommodate a rectangular mesh or an alternative discretization of Maxwell's equations on a regular grid.

III. Numerical Results

The formulation presented in the preceding section has been implemented in a finite-difference computer program. For the sake of illustrating the application of the proposed method, we consider a perfectly conducting strip grating, whose geometry, and that of the coordinate system used, are depicted in Fig. 2. The grating is illuminated by a plane wave $E_y^{\mathrm{inc}} = e^{-jk(x\sin\theta_0 + z\cos\theta_0)}$. Our objective is to determine the scattered field E_y^s. The excitation is provided by imposing the Dirichlet boundary condition $E_y^s = -E_y^{\mathrm{inc}}$ on the perfectly conducting surface of the strip. The solution domain is truncated at $z = \pm h$. The accuracy of the proposed method is verified by comparing our numerical results with the analytic solution. The error in the conservation of the power flow along the z axis, defined in [2], also serves as an internal accuracy check.

The first example considered is that of a grating with a period $d = 0.8\lambda$, and a strip width of $w = 0.4\lambda$, illuminated by a plane wave incident at an angle $\theta_0 = 0°$. This problem has been employed to test various numerical methods [9] against the analytic solution derivable by using the Wiener-Hopf technique [10]. Before proceeding to the solution of the scattering problem, it would be desirable to examine the accuracy of the NABC compared to that of the FD discretization of the Helmholtz equation. Fig. 3 presents the errors in satisfaction of the NABC (5) suffered by the forward- and backward- propagating Floquet modes, also referred to as the outgoing and incoming modes, and those incurred in the central difference approximation of the Helmholtz equation by the outgoing Floquet modes. The errors are plotted as functions of the mode order n. Here, the boundary condition is designed to absorb the outgoing Floquet modes ranging between $N_1 = -2$ and $N_2 = 2$, at $z = h = 0.2\lambda$, for a mesh

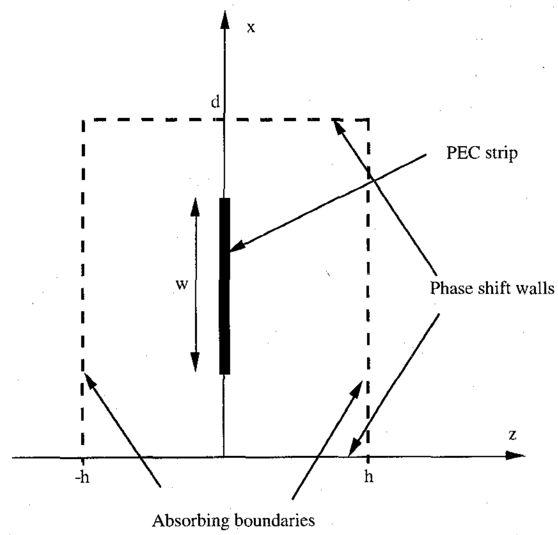


Fig. 2. A unit cell of a perfectly conducting (PEC) strip grating.

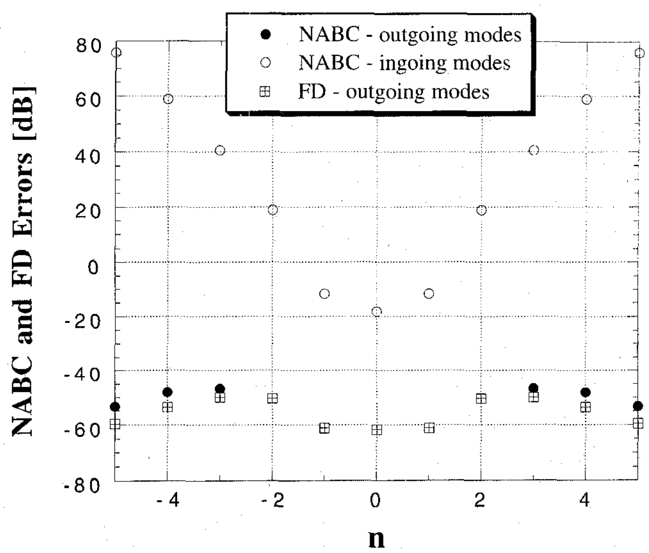


Fig. 3. Errors in satisfying the NABC and the FD scheme by various Floquet modes.

with $\Delta = 0.05\lambda$. For the outgoing modes of orders $|n| \geq 3$, the error in the NABC is comparable with the discretization error. The high errors associated with the incoming Floquet modes demonstrate the ability of the NABC to suppress the spurious solutions associated with these modes.

Fig. 4 illustrates the convergence of the finite difference solution, derived by using the NABC developed in this paper, as a function of the mesh size. Also shown in this figure is the effect of the selection of N_1 and N_2 in the boundary condition on the accuracy of the solution. We have investigated three different choices of the combination of N_1 and N_2, viz., $-N_1 = N_2 = 0, 1, 2$, for the solution domain truncated at $h = 0.2\lambda$. It is evident that for both $-N_1 = N_2 = 1$ and $-N_1 = N_2 = 2$, the computed reflection coefficient approaches the Wiener-Hopf result as the mesh size $\Delta \to 0$. In contrast, a boundary condition absorbing only the zeroth-order mode is clearly inaccurate for the truncation distance

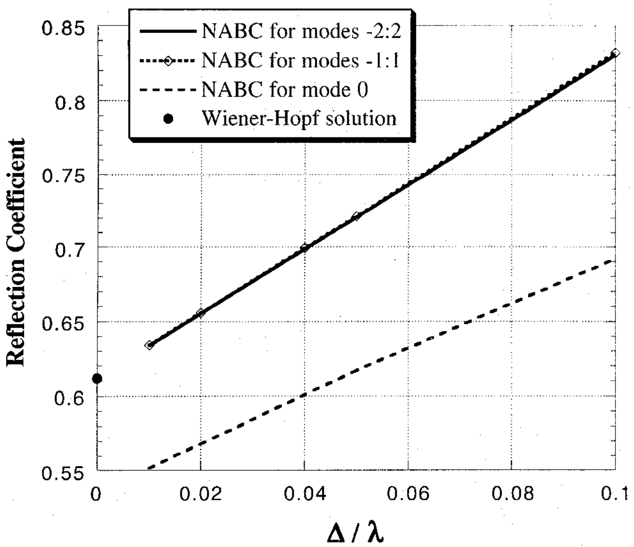


Fig. 4. Power reflection coefficient versus Δ obtained with various values of $-N_1 = N_2$ for the case of $d = 0.8\lambda$, $w = 0.4\lambda$, and $\theta_0 = 0°$ with $h = 0.2\lambda$.

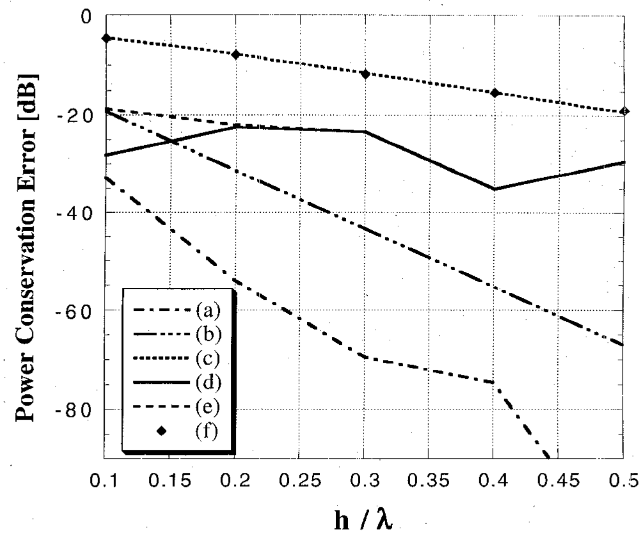


Fig. 5. Power conservation error versus h for the case of $d = 0.8\lambda$, $w = 0.4\lambda$, and $\theta_0 = 0°$ with $\Delta = 0.05\lambda$ computed with NABC using the discrete modes with (a) $-N_1 = N_2 = 2$, (b) $-N_1 = N_2 = 1$, (c) $-N_1 = N_2 = 0$ and the continuous modes with (d) $-N_1 = N_2 = 2$, (e) $-N_1 = N_2 = 1$, (f) $-N_1 = N_2 = 0$.

$h = 0.2\lambda$. Next, we consider the effect of using the 'discrete' Floquet modes with the dispersion relationship given by (9) in deriving the NABC. To this end, we compare the accuracy of the results obtained by using the NABC based on the discrete modes, with the corresponding continuous mode results. Plots of the power conservation error *vs.* h computed for three cases, viz., $-N_1 = N_2 = 0, 1, 2$, are presented in Fig. 5 for 'discrete' and 'continuous' modes. The NABC based on the discrete modes is clearly more accurate than its continuous counterpart for $-N_1 = N_2 = 1, 2$, and the choice of $-N_1 = N_2 = 2$ with discrete modes is obviously the most accurate. In fact, this choice allows the truncation distance h of the boundary of the computational domain to be as close as 0.1λ, while $-N_1 = N_2 = 1$ produces satisfactory accuracy only for $h \geq 0.2\lambda$. It is interesting to note that when more continuous modes are used, the choice of $-N_1 = N_2 = 2$ over $-N_1 = N_2 = 1$ does not improve the accuracy of the results. Clearly, the boundary condition based upon the absorption of only the zeroth-order mode (i.e. the choice $-N_1 = N_2 = 0$) is not affected by the type of Floquet mode employed, and requires the discretization of an excessively large computation domain to achieve accurate results.

In the previous example which dealt with the case of normal incidence, it was only natural to select N_1 and N_2 symmetrically around 0. We now consider a grating with a period of $d = 1.6\lambda$ and a strip width of $w = 0.8\lambda$, illuminated at $\theta_0 = 60°$. The normalized power reflection coefficient for this structure is plotted in Fig. 6 as a function of h for two combinations of (N_1, N_2), viz., (i) $N_1 = -2$, $N_2 = 2$; and (ii) $N_1 = -3$, $N_2 = 1$. Since the computational effort is the same for both of these two cases, the latter choice is the preferable one. This result could be predicted if one notes that the $n = -3$ mode is evanescent but decays relatively slowly away from the excitation, while the $n = 2$ mode is highly evanescent. Thus, it is necessary to enforce the absorption of only the former mode to obtain an efficient boundary condition, because the latter is sufficiently attenuated before reaching the truncation

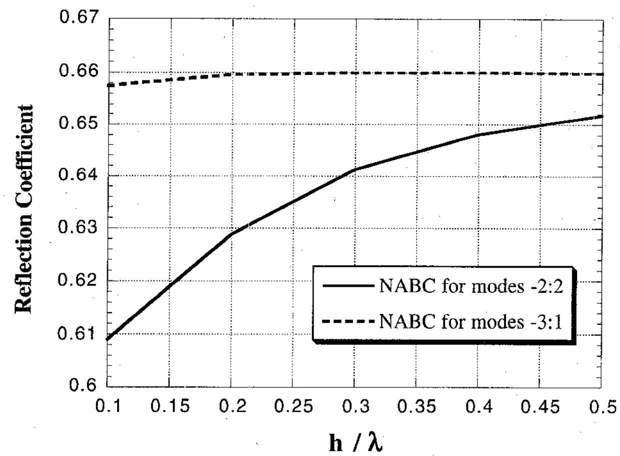


Fig. 6. Power reflection coefficient versus h for the case of $d = 1.6\lambda$, $w = 0.8\lambda$, and $\theta_0 = 60°$ computed with $\Delta = 0.02\lambda$.

boundary. On the basis of the numerical experiments, we have derived the guideline that the NABC should include all of the propagating modes and at least one evanescent mode on each side of the spectrum.

IV. Discussion

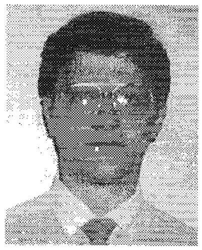
A new approach to deriving a numerical absorbing boundary condition (NABC) for mesh truncation in the FD/FEM analysis of scattering and radiation by periodic structures has been presented in this paper. The NABC is obtained by imposing the absorption condition on a selected set of Floquet modes at the truncation boundary. Unlike the boundary element method, the proposed approach preserves the sparsity of the FD/FEM matrices. The accuracy of the method has been demonstrated for a number of 2D-TM examples. Extensions of the present formulation to the TE polarization case and to the full 3D formulation are currently under investigation and the initial results look quite promising.

ACKNOWLEDGMENT

The authors are grateful to Dr. Karthik Mahadevan of the Electromagnetic Communications Laboratory of the University of Illinois at Urbana-Champaign for many fruitful discussions.

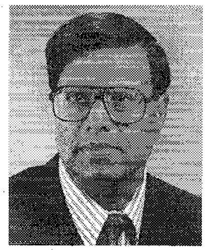
REFERENCES

[1] K. A. Zaki and A. R. Neureuther, "Scattering from a perfectly conducting surface with a sinusoidal height profile: TE polarization," *IEEE Trans. Antenn. Propagat.*, vol. AP-19, pp. 208–214, 1971.
[2] A. Boag, Y. Leviatan, and A. Boag, "Analysis of two-dimensional electromagnetic scattering from non-planar periodic surfaces using a strip current model," *IEEE Trans. Antenn. Propagat.*, vol. AP-37, pp. 1437–1446, 1989.
[3] M. K. Moaveni, H. A. Kalhor, and A. Afrashten, "An iterative finite-difference coupling technique applied to diffraction gratings," *Comput. Elect. Eng.*, vol. 2, pp. 265–271, 1975.
[4] S. D. Gedney, J. F. Lee, and R. Mittra, "A combined FEM/MoM approach to analyze the plane wave diffraction by arbitrary gratings," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-25, pp. 363–370, Feb. 1992.
[5] A. Bayliss and E. Turkel, "Radiation boundary conditions for wave-like equations," *Comm. Pure Appl. Math.*, vol. 33, pp. 707–725, 1980.
[6] B. Engquist and A. Majda, "Absorbing boundary conditions for wave-like equations," *Math. Comp.*, vol. 31, pp. 629–651, 1977.
[7] A. Bayliss and E. Turkel, "Far field boundary conditions for compressible flows," *J. Comp. Phys.*, vol. 48, pp. 182–199, 1982.
[8] K. K. Mei, R. Pous, Z. Chen and Y. W. Liu, "The measured equation of invariance a new concept in field computation," in *Proc. URSI Radio Sci. Mtg.*, Chicago, 1992.
[9] J. P. Montgomery and K. R. Davey, "The solution of planar periodic structures using iterative methods," *Electromagnetics*, vol. 5, pp. 209–235, 1985.
[10] L. A. Weinstein, *The Theory of Diffraction and Factorization Method.* Boulder, CO: Golem, ch. 8, 1969.



Amir Boag received the B Sc. degree in electrical engineering and the B A. degree in physics in 1983, both Summa Cum Laude, the M.Sc. degree in electrical engineering in 1985, and Ph.D. degree in electrical engineering in 1991, all from Technion - Israel Institute of Technology, Haifa, Israel.

From 1991 to 1992 he was on the faculty of the Department of Electrical Engineering at the Technion. In 1992 he joined the Electromagnetic Communication Laboratory of the Department of Electrical and Computer Engineering at the University of Illinois at Urbana- Champaign as a visiting assistant professor. Dr. Boag's interests are in electromagnetic theory, wave propagation and scattering, and inverse scattering problems. He has published over 20 journal papers on electromagnetics and acoustics.



Raj Mittra is the Director of the Electromagnetic Communication Laboratory of the Electrical and Computer Engineering Department and Research Professor of the Coordinated Science Laboratory at the University of Illinois. He is a Fellow of the IEEE, a Past-President of AP-S, and he has served as the editor of the Transactions of the Antennas and Propagation Society. He won the Guggenheim Fellowship Award in 1965 and the IEEE Centennial Medal in 1984. He has been a visiting professor at Oxford University, Oxford, England and at the Technical University of Denmark, Lyngby, Denmark. Currently, he serves as the North American editor of the journal AEU. He is President of RM Associates, which is a consulting organization providing services to several industrial and governmental organizations. His professional interests include the areas of electromagnetic modeling and simulation of electronic packages, EMP and EMC analysis, radar scattering, frequency selective surtaces, microwave and millimeter wave integrated circuits, and satellite antennas. He has published over 350 journal papers and 20 books or book chapters on various topics related to electromagnetics.